A GENERALIZATION OF MALHO'S METHOD FOR OBTAINING LARGE CARDINAL NUMBERS

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ABSTRACT

Mahlo used a method by which fixed points of an enumeration of regular cardinals were employed to get a hierarchy of "large cardinals." He also employed a second method which, in a certain sense, is much stronger than the first. Here the methods are investigated and generalized and the relations between them are clarified. This stronger method turns out to be a kind of "least upper bound" to all "fixed-points operations." Possibilities of strengthening these processes in a natural way are pointed out.

0. **Introduction.** In his articles from 1911–1913 (cf. [1], [2] and [3]) Mahlo uses the method of taking fixed points of an enumeration of ordinals, in order to arrive at "big ordinals." Consider an increasing sequence of ordinals $\alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots$, a fixed point would be an ordinal which is equal to its index, i.e. $\lambda = \alpha_\lambda$. Now in order that this should give us "big ordinals" it is first required that the sequence should be with "large enough" gaps. For instance, if there are no gaps at all and the sequence runs through all ordinals, then every one will be a fixed point. This, however, is not enough. Consider the sequence of all cardinals (being identified with their initial ordinals), $\omega_0, \omega_1, \cdots, \omega_\lambda, \cdots$. As it is well known there are lots of fixed points here. Just put $\phi(0) = \omega_\lambda, \phi(n+1) = \omega_{\phi(n)}$ and then $\bigcup_{n<\omega_0} \omega_{\phi(n)}$ is a fixed point. Fixed points which are obtained in this way will, as a rule, be singular and, consequently, do not merit the title "big ordinals." To avoid these "bad" fixed points, Mahlo limits himself to regular ordinals, and here, indeed, we do get something "big," because the fixed points which are regular are exactly the weakly inaccessible cardinals. The way Mahlo does it is to start from the regular ordinals, which he calls the π_0-numbers, and to take fixed points of their enumeration. This makes no difference, since the fixed points in the enumeration of the π_0-numbers are exactly the regular fixed points in the enumeration of all cardinals. These are the π_1-numbers, and the natural way to continue the process is to enumerate them in their order and take the fixed points of this enumeration, which will get us the π_2-numbers. In the same way one gets from the π_2-numbers the π_3-numbers and so on.

After continuing the process ω_0 times one gets the π_{ω_0}-numbers by taking the intersection of all the classes π_n where $n < \omega_0$. The general definition of π_α-num-

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bers, where $\alpha > 0$, can be stated, for α a limit or a non-limit ordinal, by saying that a π_α-number is a fixed point in the enumeration of the π_β-numbers, for all $\beta < \alpha$.

It should be remarked here that the existence of π_1 numbers > 0 cannot be proved; indeed, it is consistent to assume that they do not exist. Moreover, for every model M of set-theory (say, Zermelo-Fraenkel's), in which there are π_{x+1}-numbers > 0, where x and $x + 1$, are, respectively, an ordinal and its successor in M, there is a submodel N, of set-theory, which is transitive in M, having x as a member, in which there are π_x-numbers > 0 but not π_{x+1}-numbers > 0. The clause " > 0" is added here because according to Mahlo 0 is considered among the regular cardinals, the next one being ω_0. This makes 0 a π_ν-number for all ν. If one starts from ω_0 it can be omitted.

The process of taking fixed points is not finished by now. One can continue and ask whether there are ordinals ν which are π_ν-numbers. Enumerating all π_ν numbers: $\pi_{0,\nu}, \pi_{1,\nu} \cdots, \pi_{\lambda,\nu}, \cdots$, since 0 is trivially $\pi_{0,\nu}$, it can be shown that if ν is a π_ν-number then $\nu = \pi_{1,\nu}$. Thus, one can consider fixed points for the second index. Enumerating them, one can take fixed points, and continue likewise "indefinitely." A clarification of this "indefinitely" is one of the aims of the present work.

Mahlo introduces a second kind of "big ordinals." An ordinal is called by him a ρ_0-number if it is a regular ordinal and every sequence of ordinals, whose limit it is, has an initial whose limit is a smaller regular ordinal. A ρ_1-number is a ρ_0-number having the analogous property with respect to ρ_0-numbers, that is, every sequence whose limit it is has an initial whose limit is a smaller ρ_0-number. In this way the ρ_α-numbers are defined for all α, the case where α is a limit ordinal being taken care of by forming the intersection of all previously obtained classes.

It is shown by Mahlo that this second method is "stronger" than the first in the sense that every ρ_0-number is equal to some ν which is a π_ν-number. Moreover the first ν which is a π_ν number is not a ρ_0-number. The proof seems to indicate that one cannot get at the ρ_0-numbers by iterating the fixed point process. Thus every ρ_0-number is a fixed point in the enumeration of all ν's which are π_ν-numbers —and this seems to hold for any "indefinite" continuation of the fixed point process. The same situation takes place if one compares the numbers obtained from the ρ_ν-numbers by fixed point methods with the $\rho_{\nu+1}$-numbers.

It is the aim of this paper to establish the exact relationship between the two processes and to show in which sense the ρ_ν-numbers are "bigger" than those obtained by fixed points methods. It turns out that the second process can be described as the "supremum" of all iterations of the fixed point methods, and is itself a kind of "generalized" fixed point operation. This generalized process when applied, say, to the operation which determines the ρ_0-numbers does not yield the ρ_1 numbers but much "larger" ordinals whose relation to the ρ_0 numbers is perhaps more analogous to the relation which the ρ_0 numbers bear to the π_0-numbers.

Finally, the way is open to even "stronger" operations.

2. **Preliminaries.** Cardinals are identified with their initial ordinals, and every ordinal is identified with the set of all its preceeding ones.

Ord = the class of all ordinals

We will consider functions whose arguments and values are sets or proper classes of ordinals; moreover we will form classes of such functions. Thus classes of classes of classes of... to the fourth or fifth degree are used. This, howeves, is completely inessential, and is being done for convenience only. All our functions will have the property:

$$G(X \cap \alpha) = G(X) \cap \alpha$$

where G is the function, X any class of ordinals and α any ordinal. All the operations on the functions will preserve this property. The structure will have, thus, the local property that ordinals $< \alpha$ are not affected by the question whether or not our sets of ordinals include ordinals $> \alpha$. One can, therefore, limit oneself from the beginning to an initial section consisting of all ordinals $< \alpha$ and then let α be arbitrary large. In fact, the whole work can be carried in Zermelo-Fraenkel's set theory at the cost of encumbering, somewhat, the formulation.

Those who still feel unsure may imagine that all the classes involved are subsets of θ, where θ is some strongly inaccessible cardinal, and $Ord = \theta$.

For the sake of convenience we introduce, a new symbol, "∞", and make the convention that $\alpha < \infty$ for all ordinals α. ∞ is not to be considered an ordinal, and symbols such as "α", "β", $\cdots$ which are used to denote ordinals never denote ∞.

A sequence $X = \langle X_\alpha \rangle_{\alpha < \infty}$ ($\langle X_\alpha \rangle_{\alpha < \delta}$) of classes of ordinals is *decreasing* if for every $\alpha < \beta$ (every $\alpha < \beta < \delta$) $X_\alpha \supseteq X_\beta$. It is *continuously decreasing* if for every limit ordinal $\alpha > 0$ ($\delta > \alpha > 0$) we have $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$. ∞ (or δ) is the length of the sequence.

All the functions to be considered will have as arguments and values classes of ordinals. $D(f)$ is the domain of the function f.

$$f \geqq g \text{ if } D(f) = D(g) \text{ and } f(X) \supseteq g(X) \text{ for all } X \in D(f).$$

If $\{f_i\}_{i \in I} f_i$ is family of functions then $\bigcap_{i \in I} f_i$ is the function whose domain is $\bigcap_{i \in I} D(f_i)$ and whose values are given by:

$$\left(\bigcap_{i \in I} f_i\right)(X) = \bigcap_{i \in I} f_i(X)$$

$F = \langle F_\alpha \rangle_{\alpha < \infty}$ ($\langle F_\alpha \rangle_{\alpha < \delta}$) is a *decreasing sequence of functions* if for all

$$\alpha < \beta (\alpha < \beta < \delta) F_\alpha \geqq F_\beta.$$

It is *continuously decreasing* if it is decreasing and $F_\alpha = \bigcap_{\beta < \alpha} F_\beta$ for all limit ordinals (all limit ordinals $< \delta$).

If X is a decreasing sequence of classes of ordinals then X^D is defined by:

$$X^D = \{\alpha : \alpha \in X_\alpha\}\ , \text{ if} = \langle X_\alpha \rangle_{\alpha < \infty}$$

$$X^D = \{\alpha : \alpha < \delta \text{ and } \alpha \in X_\alpha\} \cup \textstyle\bigcap_{\alpha < \delta} X_\alpha, \text{ if } X = \langle X_\alpha \rangle_{\alpha < \delta}.$$

X^D is referred to as the diagonal of X.

If $F = \langle F_\alpha \rangle_\alpha$ is a decreasing sequence of functions then F^D is defined by:

$$D(F^D) = D(F_0),\ F^D(X) = Y^D, \text{ where } Y_\alpha = F_\alpha(X).$$

A function f is a *local thinning function* if it has the properties:

$X \in D(f)$ and $Y \subseteq X$ imply $Y \in D(f)$.

$f(X) \subseteq X$ for $X \in D(f)$

$f(X \cap \alpha) = f(X) \cap \alpha$, for all $X \in D(f)$ and all α, (this is equivalent to: for all β, $\beta \in f(X)$ iff $\beta \in f(X \cap (\beta + 1))$).

We will abbreviate and speak about a LTF or say that f is a LTF, $f \in LTF$.

f is said to be *monotone* if $X \subseteq Y$ implies $f(X) \subseteq f(Y)$, for all $X, Y \in D(f)$.

(Not every LTF is monotone. Consider, for example, the function which, for evry X, has as a value the set of all members of X which are not limit points of X)

The composition of two functions, $f \circ g$, is defined by:

$$D(f \circ g) = \{X : X \in D(g),\ g(X) \in D(f)\},\ (f \circ g)(X) = f(g(X)).$$

PROPOSITION 1. (i) *If* $f, g \in LTF$ *and* $D(f) = D(g)$ *then* $f \circ g \in LTF$

(ii) $\{f_i\}_{i \in I} \subseteq LTF$ *implies* $\bigcap_{i \in I} f_i \in LTF$

(iii) *If* P *is a decreasing sequence of members of* LTF *then* $F^D \in LTF$.

(iv) *The statements* (i)–(iii) *are true if* LTF *is replaced by the subclass of all the monotone functions in* LTF.

The proof is straghtforward.

Let I be the identity function ($I(X) = X$, for all X) f^α is defined by transfinite induction as follows:

$$f^0 = I \text{ restricted to } D(f)$$

$$f^{\alpha+1} = f \circ f^\alpha;\ f^\alpha = \bigcap_{\beta < \alpha} f^\beta, \text{ if } \alpha \text{ is a limit ordinal.}$$

Let $f \in LTF$, let F be the sequence $\langle f^\alpha \rangle_{\alpha \in Ord}$ then f^Δ is defined to be F^D. means that $f^\Delta(X) = \{\alpha : \alpha \in f^\alpha(X)\}$.(*)

Let X be any class of ordinals. Let $\alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots$ be the enumeration of X according to the natural order. Then $th(\lambda, X)$ is the λth ordinal of X, which is

(*) The definitions of f^α and f^Δ are due to D. Scott.

α_λ, if there is such, or ∞ if there is no such ordinal. $fp(X)$ is defined as $\{\lambda : \lambda = \alpha_\lambda\}$. $fp(X)$ is the class of the fixed points of X. We define $q(X)$ as $\{\alpha_1, \alpha_2 \cdots\}$ i.e., the class obtained by deleting the first member of X, and put $q(0) = 0$.

From proposition 1 it follows that if $f \in LTF$ then also $f^\alpha \in LTF$, for all α, and $f^\Delta \in LTF$. The same holds for monotone LTF's.

It is easily seen that:

$$fp = q^\Delta.$$

Since q is, obviously, a monotone LTF, so is fp.

The following is also easily verified.

PROPOSITION 2. (i) $X \supseteqq Y$ implies $th(\lambda, X) \leqq th(\lambda, Y)$.

(ii) If $th(\mu, X) < \infty$ then $th(\nu, X)$ is strictly increasing as a function of ν, where $\nu \leqq \mu$.

(iii) $\alpha \leqq th(\alpha, X)$

(iv) If $th(\alpha, X) > \alpha$ then $\alpha, th(\alpha, X) \notin fp(X)$

3. **The π-numbers and the ρ-numbers.** Rg = class of all regular ordinals. An ordinal α is regular if α is a limit ordinal >0 and, $\alpha \neq \bigcup_{\beta<\gamma} \alpha_\beta$ whenever $\gamma < \alpha$ and $\alpha_\beta < \alpha$ for all $\beta < \gamma$.

The members of $fp^\mu(Rg \cup \{0\})$ is what Mahlo, [1], calls the π_μ-numbers.

$$\pi_{\gamma,\mu}(X) = Df\ th(\gamma, fp^\mu(X)).$$

The following proposition sums up the properties of the doubly-indexed array $\pi_{\lambda,\mu}(X)$.

PROPOSITION 3. (i) *If* $\mu > 0$ *then* $\alpha = \pi_{\lambda,\mu}(X)$ *for some* λ, *iff for all* $\nu < \mu$ $\alpha = \pi_{\alpha,\lambda}(X)$.

(ii) $\pi_{\lambda,\nu}(X)$ *as a function of* ν *is non-decreasing.*

(iii) *If* $\mu > \nu$ *and* $\pi_{\lambda,\mu}(X) < \infty$ *then the following three conditions are equivalent* $\pi_{\lambda,\mu}(X) > \pi_{\lambda,\nu}(X)$, $\pi_{\lambda,\mu}(X) \notin fp^{\mu+1}(X)$, $\lambda \notin fp^{\mu+1}(X)$.

(iv) *If* $\mu > \lambda$ *and* $\lambda + 1 \nsubseteq X$ *then* $\pi_{\mu,\lambda}(X) \leqq \pi_{\lambda,\mu}(X)$.

(v) *Assume that* $\mu > \lambda$ *and that either* $\pi_{\lambda,\mu}(X) < \infty$ *or* $\pi_{\mu,\lambda}(X) < \infty$. *Then* $\pi_{\lambda,\mu}(X) = \pi_{\mu,\lambda}(X)$ *iff* $\mu = \pi_{\lambda,\mu}(X)$; *each side implies that* λ *is the least ordinal which is not in* X.

Proof. (i) is straightforward.

(ii) follows from the fact that $fp^\nu(X)$ is decreasing with ν, and from proposition 2 (i).

(iii) Each of the conditions $\lambda \in fp^{\mu+1}(X)$, $\pi_{\lambda,\mu} \in fp^{\mu+1}(X)$ is equivalent to $\lambda = \pi_{\lambda,\mu}(X)$. Now, $\pi_{\lambda,\mu}(X) \geqq \pi_{\lambda,\nu}(X) \geqq \lambda$. Hence $\lambda = \pi_{\lambda,\mu}(X)$ implies $\pi_{\lambda,\mu}(X) = \pi_{\lambda,\nu}(X)$. Consequently $\pi_{\lambda,\mu}(X) > \pi_{\lambda,\nu}(X)$ implies each of the other two conditions. On the other hand $\pi_{\lambda,\mu}(X) = \pi_{\lambda,\nu}(X)$ means, since $\pi_{\lambda,\mu}(X) < \infty$, that $\pi_{\lambda,\nu}(X) \in fp^{\mu}(X)$, hence $\pi_{\lambda,\nu}(X) \in fp^{\nu+1}(X)$, implying $\lambda = \pi_{\lambda,\nu}(X) = \pi_{\lambda,\mu}(X)$.

(iv) If $\pi_{\lambda,\mu}(X) = \infty$ it is obvious. Otherwise let γ be the first ordinal which is not in X. Then $\gamma \leqq \lambda$ and hence $\pi_{\nu,\mu}(X) < \infty$. Obviously $\gamma \notin fp^{\mu+1}(X)$.Hence, from (ii) and (iii) it follows that $\pi_{\gamma,\nu}(X)$ is strictly increasing as a function of ν, for all $\nu \leqq \mu$. Therefore $\pi_{\gamma,\mu}(X) \geqq \mu$, Hence $\pi_{\lambda,\mu}(X) \geqq \mu$. Put $\zeta = \pi_{\lambda,\mu}(X)$. Now $\zeta \in fp^{\mu}(X) \subseteq fp^{\lambda+1}(X)$, therefore $\zeta = \pi_{\zeta,\lambda}(X)$. Since $\zeta > \mu$ we have $\pi_{\mu,\lambda}(X) \leqq \pi_{\zeta,\lambda}(X) = \zeta = \pi_{\lambda,\mu}(X)$.

(v) If $\pi_{\lambda,\mu}(X) = \pi_{\mu,\lambda}(X) < \infty$, then, putting $\alpha = \pi_{\lambda,\mu}(X)$, it follows by (i) that $\alpha = \pi_{\alpha,\lambda}(X)$. Hence $\alpha = \mu$ and we get $\mu = \pi_{\lambda,\mu}(X)$. On the other side, if $\mu = \pi_{\lambda,\mu}(X)$, then, again by (i), $\mu = \pi_{\mu,\lambda}(X)$ and therefore $\pi_{\lambda,\mu}(X) = \pi_{\mu,\lambda}(X)$.

It is clear that if $\lambda + 1 \subseteq X$ then $\pi_{\lambda,\alpha}(X) = \lambda$, for all α. Hence, if $\pi_{\lambda,\mu}(X) = \mu > \lambda$ we have an ordinal $\leqq \lambda$ which is not in X. Let γ be the smallest one, By (iv) we have: $\pi_{\mu,\gamma}(X) \leqq \pi_{\gamma,\mu}(X)$. Since $\mu \leqq \pi_{\mu,\gamma}(X)$ one gets: $\mu \leqq \pi_{\mu,\gamma}(X) \leqq \pi_{\gamma,\mu}(X) \leqq \pi_{\lambda,\mu}(X) = \mu$. Hence $\pi_{\gamma,\mu}(X) = \pi_{\lambda,\mu}(X) < \infty$ and, consequently, $\gamma = \lambda$.

The class of fixed points for the second index is $F^{D}(X)$ where $F = \langle fp^{\nu} \rangle_{\nu < \infty}$.

We define $L(X)$, *the class of all limit points of* X, as the class of all ordinal which are of the form $\bigcup_{\lambda<\mu} \alpha_{\lambda}$, where α_{λ} is a strictly increasing sequence of members of X and μ a limit ordinal > 0. Thus $0 \notin L(X)$.

PROPOSITION 4. (MAHLO). If $X \subseteq Rg$ then $fp(X) = X \cap L(X)$.

Proof. If $\alpha = th(\alpha, X)$, then $X \cap \alpha$ forms a strictly increasing sequence of type α, α being a limit ordinal. The union of this sequence is therefore $\geqq \alpha$, on the other hand it is $\leqq \alpha$. Hence $\alpha \in L(X)$.

If $\alpha \in L(X)$ then $\alpha = \bigcup_{\lambda<\gamma} \alpha_{\lambda}$ where γ is a limit ordinal > 0 and α_{λ} a strictly increasing sequence of members of X. Since α is regular we have $\gamma = \alpha$. Put $Y = \{\alpha_{\lambda} : \lambda < \gamma\} \cup \{\alpha\}$. Then $\alpha = th(\alpha, Y)$. Since $Y \subseteq X$ it follows that if $\alpha = \mathrm{th}(\beta, X)$ then $\beta \geqq \alpha$. But we must have $\beta \leqq \alpha$. Hence $\alpha = \mathrm{th}(\alpha, X)$.

Note that the argument of the first half of the proof shows that, for all X, $fp(X) \cap L(Ord) \subseteq L(X)$.

We define X to be *closed in* Y if $L(X) \cap Y \subseteq Y$. If $X \subseteq Y$ this actually means that X is a subclass of Y which is closed in the order-induced topology. In general it means that X is a closed subclass of $X \cup Y$. A function f is *closed in* Y if, for every $X \in D(f)$, if X is closed in Y so is $f(X)$.

It is easily seen that if X_i is closed in Y for all $i \in I$ so is $\bigcap_{i \in I} X_i$. Consequently if f_i is a function which is closed in Y for all $i \in I$ so is $\bigcap_{i \in I} f_i$. If f and g are closed in Y so is $f \circ g$.

PROPOSITION 5. (i) *If* X *is continuously decreasing sequence of classes of ordinals, each of which is closed in* Y *then* X^{D} *is closed in* Y.

(ii) *If* F *is continuously decreasing sequence of functions, each closed in* Y, *then* F^{D} *is closed in* Y.

Proof. Assume with no loss of generality that $X = \langle X_\alpha \rangle_{\alpha < \infty}$, because, if $\langle X_\alpha \rangle_{\alpha < \delta}$, we can replace it by X^*, where $X^*_\alpha = X_\alpha$ for $\alpha < \delta$ and $X^*_\alpha = \bigcap_{\beta < \delta} X_\beta$ for $\alpha > \delta$. Then $X^D = (X^*)^D$ and each X^*_α is closed in Y.

Let $\alpha \in L(X^D) \cap Y$. Then $\alpha = \bigcup_{\gamma < \delta} \alpha_\gamma$, where $\langle \alpha_\gamma \rangle_{\gamma < \delta}$ is a strictly increasing sequence of members of X^D and δ is a limit ordinal. $\alpha_\gamma \in X_{\alpha_\gamma}$ for all $\gamma < \delta$. Hence $\alpha_\beta \in X_{\alpha_\gamma}$ for all $\gamma < \beta < \delta$. Consequently $\alpha \in L(X_{\alpha_\gamma})$ for all $\gamma < \delta$, and, since X_{α_γ} is closed in Y, it follows that $\alpha \in X_{\alpha_\gamma}$ for all $\gamma < \delta$. Therefore $\alpha \in \bigcap_{\gamma < \delta} X_{\alpha_\gamma}$, but since X is continuously decreasing this intersection is X_α. Thus, $\alpha \in X_\alpha$, i.e. $\alpha \in X^D$.

(ii) follows from (i).

Proposition 6. *If $X \subseteq Rg$ and X is closed in Rg then $fp(X \cap L(X)) \cap Rg = fp(X)$.*

The proof follows the usual considerations. The function $fp(X \cup L(X)) \cap In$, where In is the class of inaccessible cardinals was used by Lévy in [4]. This amounts exactly to $fp(X)$ for closed subclasses of In.

Definition. $h(X)$ is the class of all limit ordinals, $\alpha \in X$, such that, for every $Y \subseteq \alpha$, if $\alpha \in L(Y)$ then $L(\gamma) \cap X \cap \alpha \neq 0$.

The function h, applied to Rg and iterated, was first used by Mahlo.

If $\alpha \in h(X)$ then $\alpha \in L(X)$, because otherwise, for some $\beta < \alpha$, we will have $\{\xi : \beta < \xi < \alpha\} \cap X = 0$, contradicting the requirement for $Y = \{\xi : \beta < \xi < \alpha\}$.

D. Scott suggested the version: $\alpha \in h(X)$ if it is a limit ordinal and, for all $Y \subseteq \alpha$, if Y is closed in α and $\alpha \in L(Y)$ then $Y \cap X \neq 0$. This version is equivalent to the one given here for $X \subseteq L(Ord)$. In general we have $h_s(X) \subseteq h(X)$, if h_s is the function as defined by Scott.

Obviously h is a monotone LTF.

It can be easily seen that, for $X \subseteq Rg$, $h(X) \subseteq fp(X)$. (Letting $\{\alpha_\lambda\}_\lambda$ be the enumeration of X, if $\alpha = \alpha_\lambda > \lambda$ consider $\beta = \bigcup_{\nu < \lambda} \alpha_\nu$ and $Y = \{\xi : \beta < \xi < \alpha\}$.)

Theorem (Mahlo). (I) *If $Y \subseteq Rg$ and $\pi_{\mu,\nu}(Y) > \mu, \nu$ then $\pi_{\mu,\nu}(Y) \notin h(Y)$.* (II) *If $Y \subseteq Rg$ then the first $\alpha > 0$ such that $\alpha \in fp^\alpha(Y)$ does not belong to $h(Y)$.*

This follows from the generalization which we formulate and prove next.

The idea of the following theorems is to indicate a sense in which the function h is "stronger" than other "decent" functions. Hereby, "stronger" roughly means that its application yields smaller classes of ordinals, whose members can, therefore, be considered as "larger." "Decent" includes, among the rest, the function fp, as well as any function fp, fp^Δ, and lots of others which are specified in this section.

Theorem 1 implies that, under certain conditions, a regular α, which is in $h(\gamma)$, cannot be removed by an application of any function which arises out of a "decent function" by means of a process which involves composition of functions, iterations of less than α times, and forming the diagonal of a continuously decreasing sequence of functions.

Mahlo's theorem, which is implied by Theorem 1, deals with two special cases which are typical to the general state of affairs. The first case is that where $\alpha = \pi_{\mu,\nu}(Y) > \mu, \nu$. Here, applying fp^{μ} will still leave α, but one more application of fp (or $\mu + 1$ applications of q) will remove it. In the second case α is the first β such that $\beta = fp^{\beta}(Y)$. Here an application of the diagonal of $\langle fp^{\beta}\rangle_{\beta<\alpha}$ will leave α, but one more application of fp (or q) will remove it. The conclusion is that in both cases $\alpha \notin h(Y)$.

4. The general method.

THEOREM 1. *Let f be a LTF such that every member of $D(f)$ is a subclass of Y, and, for all $X \in D(f)$,*

$$f(X) \cap L(Y) = L(X) \cap X.$$

Let $\alpha \in h(Y) \cap Rg$.

(I) If $Z \in D(f)$ is closed in Y, then $\alpha \in f(Z)$ implies $\alpha \in f(f(Z))$

(II) If $0 < \beta < \alpha$ and, for all $\gamma < \beta$, $X_\gamma \in D(f)$, X_γ is closed in Y and $\alpha \in f(X_\gamma)$ then $\alpha \in f(\cap_{\gamma<\beta} X_\gamma)$.

(III) If $X = \langle X_\gamma\rangle_{\gamma<\delta}$ is a continuously decreasing sequence and, for all $\gamma < \alpha, \delta$, we have $\alpha \in f(X_\gamma)$, then $\alpha \in f(X^D)$.(*)

(The theorem can be generalized by omitting the requirements that f is a *LTF* and that $D(f)$ consists of subsets of Y. The requirements which have to be made are:

$$f(X) \supseteq X \cap L(X) \cap Y \text{ and } L(X) \supseteq f(X) \cap L(Y).$$

In (I), (II), (III) one has to add in every place the condition that the classes in question are in $D(f)$. The proof is the same).

Proof. The ideas are the same as in the proof given by Mahlo for the previous theorem.

(I) In order to show $\alpha \in f(f(Z))$ it suffices to show $\alpha \in L(f(Z))$. Since $f(Z) \supseteq Z \cap L(Z)$ it is enough to show $\alpha \in L(Z \cap L(Z))$. Now $\alpha \in f(Z)$, hence $\alpha \in L(Z)$. Since $\alpha \in h(Y)$, α is not confinal with ω_0. This is easily seen to imply $\alpha \in L(L(Z))$. Consequently, for every $\beta < \alpha$, $\alpha \in L([\beta,\alpha) \cap L(Z))$, where $[\beta,\alpha) = \{\xi : \beta \leqq \xi < \alpha\}$. Since $[\beta,\alpha) \cap L(Z)$ is closed in α it follows that $[\beta,\alpha) \cap L(Z) \cap Y \neq 0$. But Z is closed in Y, hence $L(Z) \cap Y \subseteq Z$. Consequently, for every $\beta < \alpha$, $[\beta,\alpha) \cap Z \cap L(Z) \neq 0$, which proves that $\alpha \in L(Z \cap L(Z))$

(II) By induction on β. For $\beta = 1$ -obvious. If $\beta = \delta + 1$ and the claim holds for $\beta = \delta$ then $\alpha \in f(\bigcap_{\gamma<\delta} X_\gamma)$. Putting $Z_1 = \bigcap_{\gamma<\delta} X_\gamma$ and $Z_2 = X_\delta$ it suffices to

2. The theorem is also true if we use Scott's definition of h. Indeed, the same proof is valid.

show that $\alpha \in f(Z_1 \cap Z_2)$, from the assumptions that $\alpha \in f(Z_i)$ $i = 1, 2$, and Z_i are closed subclasses of Y. Now $\alpha \in L(Z_1) \cap L(Z_2)$. Hence, given $\beta < \alpha$, one can construct a sequence $\xi_0 < \zeta_0 < \xi_1 < \zeta_1 < \cdots < \xi_n < \zeta_n < \cdots$ where $\xi_i \in Z_1, \zeta_i \in Z_2$, $\xi_0 > \beta$ and $\xi_n, \zeta_n < \alpha$. Then $\bigcup_{n<\omega_0} \zeta_n = \bigcup_{n<\omega_0} \zeta_n$ and it is $< \alpha$, since α is not confinal with ω_0. This shows that α is a limit point of $L(Z_1) \cap L(Z_2)$, and is, consequently, a limit point of $[\beta, \alpha) \cap L(Z_1) \cap L(Z_2)$, for all $\beta < \alpha$. Since $[\beta, \alpha) \cap L(Z_1) \cap L(Z_2)$ is closed in α, its intersection with Y is non empty. Hence $\alpha \in L(Y \cap L(Z_1) \cap L(Z_2))$. But $Y \cap L(Z_i) \subseteq Z_i$, $i = 1, 2$, the Z_i being closed. Therefore $\alpha \in L(Z_1 \cap Z_2)$. Since $\alpha \in Z_1 \cap Z_2$ this implies $\alpha \in f(Z_1 \cap Z_2)$. This takes care of the passage from δ to $\delta + 1$.

Assume (II) to hold for all $\beta < \delta$ where $\delta < \alpha$ is a limit ordinal. Given $\langle X_\gamma \rangle_{\gamma<\delta}$, put $Z_\gamma = \bigcap_{\lambda<\gamma} X_\lambda$. Obviously $\bigcap_{\gamma<\delta} Z_\gamma = \bigcap_{\gamma<\delta} X_\gamma$. By our assumption $\alpha \in f(Z_\gamma)$ and hence $\alpha \in L(Z_\gamma)$, for all $\gamma < \delta$. Moreover, the Z_γ form a decreasing sequence of classes which are closed in Y. Let β be any ordinal $< \alpha$. Let $\alpha_{0,0}$ be the first ordinal in Z_0 which is $> \beta$. In general let $\alpha_{0,\gamma}$ be the first ordinal in Z_γ which is $> \bigcup_{\lambda<\gamma} \alpha_{0,\lambda}$. Here we use the fact that α is regular to infer that $\bigcup_{\gamma<\delta'} \alpha_{0,\gamma} < \alpha$, for all $\delta' < \delta$. (This is actually proved by induction on δ'. If it is true for δ' it is true for $\delta' + 1$, because $\alpha \in L(Z_{\delta'})$, and for limit ordinals it is true because they are $< \alpha$ and α is regular.) Now continue to define $\alpha_{1,0}$ as the first member of Z_0 which is $> \bigcup_{\gamma<\delta} \alpha_{0,\gamma}$, $\alpha_{1,\gamma}$ as the first member of Z_γ which is $> \bigcup_{\lambda<\gamma} \alpha_{1,\lambda}$ and so on. One gets in this way a double sequence $\langle \alpha_{\eta,\gamma} \rangle_{\eta<\alpha,\gamma<\delta}$ having the properties: $\beta < \alpha_{0\,0}$

$$\alpha_{\eta\,\gamma} \in Z_\gamma \text{ and } \bigcup_{\lambda<\gamma} \alpha_{\eta,\lambda} < \alpha_{\eta.\gamma} < \alpha$$

$$\alpha_{\eta,0} > \bigcup_{\zeta\,<\eta,\gamma<\delta} \alpha_{\zeta,\gamma}$$

The regularity of α insures that all the $\alpha_{\eta,\gamma}$ are $< \alpha$.

Put $\beta_\eta = \bigcup_{\gamma<\delta} \alpha_{\eta,\gamma}$. Then $\beta_\eta \in L(Z_\gamma)$ for every $\gamma < \delta$, and $\bigcup_{\eta<\alpha} \beta_\eta = \alpha$. Let $T = \{\beta_\eta : \eta < \alpha\} \cup L(\{\beta_\eta : \eta < \alpha\})$. Since T is closed in α and $\alpha \in L(T)$ we have $T \cap Y \neq 0$. But $\xi \in T$ implies $\xi \in L(Z_\gamma)$, and $L(Z_\gamma) \cap Y \subseteq Z_\gamma$, therefore $T \cap Y \subseteq \bigcup_{\gamma<\delta} Z_\gamma$. Thus, $[\beta, \alpha) \cap \bigcap_{\gamma<\delta} Z_\gamma \neq 0$ for every $\beta < \alpha$, implying $\alpha \in L(\bigcap_{\gamma<\delta} Z_\gamma)$. Since $\alpha \in \bigcap_{\gamma<\delta} Z_\gamma$ we have $\alpha \in f(\bigcap_{\gamma<\delta} Z_\gamma)$.

(III) Since, for all $\gamma < \alpha, \alpha \in f(X_\gamma)$ we have $\alpha \in X^D$. It suffices to show $\alpha \in L(X^D)$. With no loss of generality it can be assumed that the length of X is α. For if $X = \langle X_\gamma \rangle_{\gamma<\delta}$ and $\delta < \alpha$, then $X^D \supseteq \bigcap_{\gamma<\delta} X_\gamma$, and by (II) $\alpha \in L(\bigcap_{\gamma<\delta} X_\gamma)$.

Let $\beta < \alpha$. Let α_0 be the first members of X_0 which is $> \beta$. Let α_1 be the first member of X_{α_0} which is $> \alpha_0$. In general, let $\alpha_{\lambda+1}$ be the first member of X_{α_λ} which is $> \alpha_\lambda$, and, if λ is a limit ordinal, let $\alpha_\lambda = \bigcup_{\gamma<\lambda} \alpha_\gamma$. The regularity of α implies that $\alpha_\lambda < \alpha$ for all $\lambda < \alpha$. The set $\{\alpha_\lambda\}_\lambda$, where λ ranges over all limit ordinals $< \alpha$, has α as its limit and is closed in α. Therefore, for some arbitrary large limit ordinal $\lambda < \alpha$, we have $\alpha_\lambda \in Y$. Now, $\alpha_\lambda = \bigcup_{\gamma<\lambda} \alpha_{\gamma+1}$ and $\alpha_{\gamma+1} \in X_{\alpha_\gamma}$. Hence $\alpha_\lambda \in L(X_{\alpha_\gamma})$

for all $\gamma < \lambda$. Since X_{α_γ} is closed in Y we have $\alpha_\lambda \in X_{\alpha_\gamma}$, for all $\gamma < \lambda$. Consequently $\alpha_\lambda \in \bigcap_{\gamma<\lambda} X_{\alpha_\lambda} = X_{\alpha_\lambda}$. Therefore $\alpha_\lambda \in X^D$, which shows that $\alpha \in L(X^D)$.

To deduce Mahlo's theorem from Theorem 1, assume that $\alpha \in h(Y)$, where $Y \subseteq Rg$. The restriction of fp to subclasses of Y satisfies the conditions which are imposed on f in Theorem 1. We know also that $h(Y) \subseteq fp(Y)$. Applying (I) of Theorem 1, one deduces that $\alpha \in fp^2(Y)$,$\alpha \in fp^3(Y)$, and so on. Using (I) together with (II) one deduces that $\alpha \in fp^{\nu+1}(X)$ for all $\nu < \alpha$. Consequently $\alpha = \pi_{\alpha,\nu}(X)$ for all $\nu < \alpha$. This implies the first part of Mahlo's theorem. To get the second part, consider $X = \langle X_\lambda \rangle_{\lambda<\alpha}$ where $X_\lambda = fp^\lambda(Y)$. Since $\alpha \in fp(X_\lambda)$ for all $\lambda < \alpha$, we have, by (III), $\alpha \in fp(X^D)$. Hence α cannot be the first member of X^D.

DEFINITION. For $f \in LTF$ define $Q(\alpha,f)$ as the smallest class, E, which satisfies:

(i) f and the restriction of I to $D(f)$ are in E.

(ii) $g \in E$ implies $f \circ g \in E$.

(iii) If $\beta < \alpha$ and $F = \langle F_\lambda \rangle_{\lambda<\beta}$ is a sequence of members of E then $\bigcap_{\lambda<\beta} F_\lambda \in E$.

(iv) If F is a continuously decreasing sequence of members of E whose length is α, if α is a limit ordinal, and $\alpha + 1$ if α is not a limit ordinal, then $F^D \in E$.

There are classes which satisfy these conditions, e.g., the class LTF. The intersection of all of these is $Q(\alpha,f)$.

One can replace (iv) by the stronger condition:

(iv′) If F is a continuously decreasing sequence of members of E then $F^D \in E$.

It makes no difference to the statements which follow. This is so because:

PROPOSITION 7. (i) *If X is a decreasing sequence of classes of ordinals, of length $> \alpha$ and Y is the sequence of length α defined by: $Y_\lambda = X_\lambda \cap (\alpha + 1)$, for all $\lambda < \alpha$,* then $Y^D \cap \alpha = X^D \cap \alpha$.

(ii) *If in* (i) *α is a limit ordinal and X is continuously decreasing then also $Y^D \cap (\alpha + 1) = X^D \cap (\alpha + 1)$*

(iii) *If F is a decreasing sequence of LTF's, of length $> \alpha$ and G is the initial segment of F of length α, then for all $X \in D(F^D)$, $F^D(X) \cap \alpha = G^D(X) \cap \alpha$.*

(iv) *If α is a limit ordinal and F is continuously decreasing then $F^D(X) \cap (\alpha+1) = G^D(X) \cap (\alpha + 1)$.*

(iii) follows from (i) and (ii) whose proof is straightforward.

If $Q'(\alpha,f)$ is the class obtained by replacing (iv) by (iv′), and if $g \in Q'(\alpha,f)$ then the restriction of g to subsets of $\alpha + 1$ is in $Q(\alpha,f)$. This is shown by proving, with the help of Proposition 8, that the class of all LTF's whose restrictions are in $Q(\alpha,f)$ satisfies the conditions of $Q'(\alpha,f)$.

PROPOSITION 8. (i) *$Q(\alpha,f) \subseteq LTF$ and all its members have the same domain*

(ii) *$h,g \in Q(\alpha,f)$ implies $h \circ g \in Q(\alpha,f)$.*

The proof (i) is obvious. (ii) is proved by showing that, for any given $g \in Q(\alpha,f)$, the class of all functions h which satisfy $h \circ g \in Q(\alpha,f)$ satisfies the conditions of the definition of $Q(\alpha,f)$.

DEFINITION. $J_{\alpha,f}(X) = \bigcap \{g(X): g \in Q(\alpha,f)\}, \ X \in D(f)\}$.
$f^{\nabla}(X) = \{\alpha: \alpha \in J_{\alpha,f}(X)\}$.

Thus $f^{\nabla} = J_f^D$ where $J_f = \langle J_{\alpha f}\rangle_{\alpha \in Ord}$

The remark following proposition 8 implies that f^{∇} would have been the same if, in the definition of $Q(\alpha,f)$, (iv) is replaced by (iv′).

The operation $f \to f^{\nabla}$ can be described as the "supremum" of all the iterated diagonal operations. To see how "strong" f^{∇} is, let α be any ordinal > 0. Let f be a LTF and let g be its restriction to subsets of $\alpha + 1$.

We will have $g^{\beta} \in Q(\alpha, g)$ for all $\beta < \alpha$. Hence, since the domain of g is limited to subsets of $\alpha + 1$ and $\langle g^{\beta}\rangle_{\beta<\alpha}$ is continuously decreasing, we have $g^{\Delta} \in Q(\alpha, g)$ Going on, this implies that $(g^{\Delta})^{\Delta} \in Q(\alpha,g)$ and so on. If $g^{\Delta\beta}$ is defined by: $g^{\Delta\beta+1} = (g^{\Delta\beta})^{\Delta} \cap g^{\Delta\beta}$ and $g^{\Delta\lambda} = \bigcap_{\gamma<\lambda} g^{\Delta\gamma}$, for limit ordinals λ, then $g^{\Delta\beta} \in Q(\alpha,f)$ for all $\beta < \alpha$. If g^* is obtained by diagonalizing over this sequence, then since the sequence is decreasing continuously, $g^* \in Q(\alpha,f)$. One can continue on to form $(g^*)^*, \cdots$, diagonalize over this, etc. As long as we diagonalize over sequences which are decreasing continuously we are still in $Q(\alpha,g)$. Therefore $g^{\nabla}, g^* \cdots$ are all $\geqq g^{\nabla}$. Since this is true for all α we have: $f^{\Delta} \geqq f^{\nabla}, f^* \geqq f^{\nabla} \cdots$ etc.

f^{∇} is also defined by a diagonal process. However, this is different from the diagonalizations used in defining the members of $Q(\alpha,f)$. In $Q(\alpha,f)$ we allowed for diagonalizations over continuously decreasing sequences of functions. Now $\langle J_{\alpha,f}\rangle_{\alpha \in Ord}$ is, in general, not continuously decreasing. Take for example $f = q$. It is not difficult to show that $q^{\nabla}(Ord) = Rg - \{\omega_0\}$ and $Rg - \{\omega_0\}$ is not closed in Ord. On the other hand q is closed, and hence, using Proposition 4, it follows that every member of $Q(\alpha q)$ is closed. Consequently the $J_{\alpha,q}$'s form a sequence of closed functions. If it were continuously decreasing then q^{∇} would have been closed. which it is not.

All the diagonal operations on continuously decreasing sequence preserve the property of being closed. The fact that q^{∇} is not so indicates that here we made indeed a jump.

THEOREM 2. (I) *If* $f \in LTF$ *is closed in* Y, $Y \in D(f)$, *and, for every* $X \subseteq Y$, $f(X) \cap L(Y) = L(X \cap X$, then $h(Y) \cap Rg \subseteq f^{\nabla}(Y)$.

(II) *For every* $Y \subseteq Rg$, $h(Y) \subseteq f p^{\nabla}(Y)$

Proof. (I) Let $\alpha \in h(Y) \cap Rg$. We claim that, for every $g \in Q(\alpha,f)$, $\alpha \in f(g(Y))$, hence a fortiori $\alpha \in g(Y)$.

Put $T = \{g: g$ is closed in Y and $\alpha \in f(g(Y)\}$. Since $\alpha \in L(Y) \cap Y$, we have $\alpha \in f(Y) = f(I(Y))$. Thus, $I \in T$. By Theorem 1 (I), putting $Z = Y$ we get $\alpha \in f(f(Y))$. Hence $f \in T$. If $g \in T$ then $\alpha \in f(g(Y))$. Putting $Z = g(Y)$ we get, by Theorem 1 (I), $\alpha \in f(f(g(Y)))$.

The other conditions in the definition of $Q(\alpha,f)$ are also satisfied by T. If, for all $\gamma < \beta$, F_γ, is a member of T and $\beta < \alpha$, then we deduce from Theorem 1(II), by putting $X_\gamma = F_\gamma(Y)$, that $\bigcap_{\gamma<\beta} F_\gamma \in T$. The last condition is deduced in a similar way. Hence $T \supseteq Q(\alpha,f)$. Consequently $\alpha \in g(Y)$ for all $g \in Q(\alpha,f)$, implying $\alpha \in f^{\nabla}(Y)$.

(II). By Proposition 4, we have, for every $X \subseteq Y$, $fp(X) = X \cap \{L(X) \cup \{0\}\}$, implying $fp(X) \cap L(Y) = X \cap L(X)$. Then apply (I).

q.e.d.

PROPOSITION 9. *If $f \leqq g$ and both are monotone LTF's then $f^{\nabla} \leqq g^{\nabla}$.*

Proof. One shows that for every $h \in Q(\alpha,g)$ there is $h' \in Q(\alpha,f)$ such that $h' \leqq h$. This is done by proving that the class of functions h for which such an h' exists satisfies the conditions of $Q(\alpha,g)$.

If $0 \notin X$ then $fp(X) \leq q(X)$. Hence, by Theorem 2 we have $h(X) \subseteq q^{\nabla}(X)$ for every $X \subseteq Rg$. Actually, equality holds.

THEOREM 3. *For every X, $q^{\nabla}(X) \subseteq h(X)$.*

Proof. Assume $\alpha \notin h(X)$ and show $\alpha \notin q^{\nabla}(X)$. First let α be a limit ordinal > 0. Let $Y \subseteq \alpha$ be such that $\alpha \in L(Y)$ and $X \cap L(Y) = 0$. Enumerate $Y \cup L(Y)$ in the natural order: $\alpha_0, \alpha_1, \cdots, \alpha_\lambda, \cdots, \lambda < \delta$. Then for limit ordinals $\lambda > 0$ we have $\alpha_\lambda = \bigcup_{\gamma<\lambda} \alpha_\lambda$ and $\alpha_\lambda \notin X$. Define: $f_\lambda = q^{\alpha_\lambda+1}$, for λ a non limit ordinal or 0, and $f_\lambda = q^{\alpha_\lambda}$ otherwise. Then $f_\lambda \in Q(\alpha,q)$ for all $\lambda < \delta$. If $\delta < \alpha$ then $q^{\alpha} = \bigcap_{\lambda<\delta} f_\lambda \in Q(\alpha,q)$ hence $q^{\alpha+1} \in Q(\alpha,q)$. But $\alpha \notin q^{\alpha+1}(X)$ hence $\alpha \notin q^{\nabla}(X)$. If $\delta = \alpha$ then if $\lambda < \alpha_\lambda$ is a non limit ordinal or 0 we have $\lambda < \alpha_\lambda < \alpha_{\lambda+1}$, hence $\lambda \notin f_\lambda(X)$. Otherwise $\lambda \notin X$ and a fortiori $\lambda \notin f_\lambda(X)$. Consequently $F^D(X) \cap \alpha = 0$ where $F = \langle f_\lambda \rangle_{\lambda<\alpha}$. This implies $\alpha \notin q(F^D(X))$. Since $q \circ F^D \in Q(\alpha,q)$ it follows that $\alpha \notin q^{\nabla}(X)$.

If $\alpha = 0$ the claim is obvious. If α is a non-limit ordinal > 0, let $\alpha = \beta + k$, where β is a limit ordinal. Then $q^{\beta} = \bigcap_{\gamma<\beta} q^{\gamma} \in Q(\alpha,q)$. Hence, applying (i) of the definition $k+1$ times, we have $q^{\beta+k+1} \in Q(\alpha,f)$. But $\alpha \notin q^{\alpha+1}(X)$.

q.e.d.

Thus for $X \subseteq Rg$ we have:

$$h(X) = fp^{\nabla}(X) = q^{\nabla}(X).$$

It seems that the "jump" from q to h amounts to the ∇ operation. Thus, if one wishes to make such another jump, the natural way would be not to iterate h but to take h^{∇}. One can of course start iterating the ∇ operation take fixed points etc. In this way one gets a new operation, ∇^*, which is defined similarly to ∇. The difference is that in this definition $Q^*(\alpha,f)$ is required also to be closed under ∇, i.e. $g \in Q^*(\alpha,f)$ should imply $g^{\nabla} \in Q^*(\alpha,f)$. One can still continue on in this direction, getting stronger and stronger operations and adding them to the set of operations which is used to define $Q(\alpha,f)$ but here we prefer to stop.

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